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Combinatorial Identities of Engelberg and Jensen's Formula

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1. INTRODUCTION

Making use of the operator $T_k = x(k + xD)$, where k is a constant, [15] we obtained the operational generating relation

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{b+1+mn}\{f(x)\} = \frac{(1+v)^{b+1}}{1-mv} f[x(1+v)], \tag{1.1}$$

where $f(x)$ admits a formal power series expansion in x, $v = xt(1+v)^{m+1}$, m and b are constants. Also [16], we proved that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^{n-1}_{a+1+mn}\{af(x) + xf'(x)\} = \frac{(1+v)^a}{x} f[x(1+v)], \tag{1.2}$$

where $v = xt(1+v)^{m+1}$.

Gould, in a series of papers, considered convolution identities and proved [4, 5] the generalized Vandermonde's convolution (2.1). Also, Gould [6, 7] and Carlitz [1, 2] considered Jensen's formula (3.1) and Gould and Kaucky [9] proved Engelberg's identities (4.1) and (4.2). Again, Gould [4, 5, 8, 9, 10, 12] made a study of the Rothe–Hagen formula (5.1) and Gould [11, p. 52] conjectured that a proof of (5.1) may be given by using only the relation

$$\sum_{k=0}^{n} \binom{a+bk}{k}\binom{c+(n-k)\,b}{n-k} = \sum_{k=0}^{n} \binom{a+d+bk}{k}\binom{c-d+(n-k)\,b}{n-k}. \tag{1.3}$$

In this paper, by direct applications of the operator formulas in (1.1) and (1.2), we propose to derive above results. To this end, in Section 2, we derive (2.1) and, in Section 3, we obtain a generalization of Jensen's formula. In Section 4, we obtain generalizations of the identities of Engelberg and, in Section 5, we establish Gould's conjecture.

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2. Convolution Formula

Gould [4, 5] obtained the generalized convolution identity

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\binom{c+(n-k)b}{n-k} = \binom{a+c+bn}{n}. \tag{2.1}$$

It is interesting to note that (2.1) follows directly from (1.1) and (1.2). In fact, from (1.1) and (1.2), we have

$$\begin{aligned} x &\sum_{n,k=0}^{\infty} \frac{t^{n+k}}{n!k!} T^{n}_{c+1+bn}\{f(x)\}\, T^{k-1}_{a+1+bk}\{ag(x)+xg'(x)\} \\ &= \frac{(1+v)^{a+c+1}}{1-bv} f[x(1+v)]\, g[x(1+v)] \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} T^{n}_{a+c+1+bn}\{f(x)\, g(x)\}, \end{aligned} \tag{2.2}$$

where $v = xt(1+v)^{b+1}$. Putting $f(x) = g(x) =$ constant, in (2.2), comparing the coefficients of t^n on both sides, the result in (2.1) follows.

3. Jensen's Formula

Carlitz [1, 2] and Gould [6, 7] considered Jensen's formula [13]

$$\sum_{k=0}^{n} \binom{a+bk}{k}\binom{c-bk}{n-k} = \sum_{k=0}^{n} \binom{a+c-k}{n-k} b^k. \tag{3.1}$$

Earlier [14], we proved that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_a{}^n\{f(x)\} = (1-xt)^{-a} f\left(\frac{x}{1-xt}\right), \tag{3.2}$$

where $f(x)$ admits a formal power series expansion in x. In order to derive (3.1), we consider the repeated operation

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-bk-k} T^{k}_{a-k+1} \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n x^{-n} T^{n}_{-c}\{x^{bk}\} \right\}. \tag{3.3}$$

Using (3.2) and the property $T_a{}^k\{x^b\} = x^b T^{k}_{a+b}\{1\}$ of the operator, we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-bk-k} T^{k}_{a-k+1} \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n x^{-n} T^{n}_{-c}\{x^{bk}\} \right\} \\ &= (1+t)^c \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T^{k}_{a+bk-k+1}\{1\}. \end{aligned}$$

Now, since $T_k^n\{x^b\} = (b + k)_n x^{b+n}$, making use of (1.1), we have Jensen's formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{a+bk}{k} \binom{c-bk}{n-k} t^n = (1+t)^c \frac{(1+v)^{a+1}}{1-(b-1)v}, \tag{3.4}$$

where $v(1+t)^b = t(1+v)^b$, b is a constant. The result in (3.4) is the same as (3.1). In fact, since $v(1+t)^b = t(1+v)^b$, for all t, we conclude that $v = t$, and the result in (3.4) is easily seen to reduce to (3.1).

The above considerations lead us to a generalization of Jensen's formula. Indeed, to obtain a generalization, we consider, in view of (3.3), a general repeated operation

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-bk-k} T^k_{a-mk+1} \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n x^{-n} T^n_{-c}\{x^{bk}\} \right\}. \tag{3.5}$$

Proceeding as before, we have, by using (1.1) and (3.2), the general result

$$\sum_{n,k=0}^{\infty} \binom{a+k-mk+bk}{k} \binom{c-bk}{n} t^{n+k} = (1+t)^c \frac{(1+v)^{a+1}}{1-(b-m)v}, \tag{3.6}$$

where $v(1+t)^b = t(1+v)^{b-m+1}$, b and m are constants. The above result reduces to (3.4) for $m = 1$.

4. Identities of Engelberg

Through probability considerations, Engelberg [3] obtained the combinatorial identities

$$\sum_{k=0}^{n} \frac{a-n}{a-n+2k} \binom{a-n+2k}{k} \binom{2n-2k}{n-k} = \binom{a+n}{n} \tag{4.1}$$

and

$$\sum_{k=0}^{n-1} \frac{1}{1+bk} \binom{k(b+1)}{k} \binom{(n-k)(b+1)-2}{n-1-k} = \frac{1}{b+1} \binom{n(b+1)}{n}, \tag{4.2}$$

and Gould and Kaucky [9] gave alternative proof of (4.1) and (4.2) and obtained their generalizations. It is interesting to note that (4.1) and (4.2) are a direct consequence of the operator formulas in (1.1) and (1.2) and admit natural generalizations.

Putting $f(x) =$ constant, in (1.1) and (1.2), we observe that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{b+nb}\{1\} \sum_{k=0}^{\infty} \frac{t^k}{k!} x T^{k-1}_{a+1+kb}\{a\} = \frac{(1+v)^{a+b}}{1-bv} = \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{a+b+nb}\{1\}, \tag{4.3}$$

where $v = xt(1+v)^{b+1}$, b is a constant. Using the definition of the operator and comparing the coefficients of t^n, in (4.3), we immediately get the identity

$$\sum_{k=0}^{n} \frac{a}{a+kb} \binom{a-1+(b+1)k}{k} \binom{b-1+(b+1)(n-k)}{n-k} = \frac{n+1}{a+b+(b+1)n} \binom{a-1+(b+1)(n+1)}{n+1} \tag{4.4}$$

which, for $a = 1$, reduces to the Engelberg's result (4.2).

To prove (4.1), we use the operational generating formula [14]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{a+n}^{n}\{f(x)\} = (1-4xt)^{-1/2} \left[\frac{2}{1+(1-4xt)^{1/2}}\right]^{a-1} f\left[\frac{2x}{1+(1-4xt)^{1/2}}\right]. \tag{4.5}$$

Now, since

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T_a^{k} \left\{\sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{b+1+n}^{n}\{f(x)\}\right\} \\ &= (1-t)^{-a} \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} (1-t)^n T_{b+1+n}^{n}\{f(x)\} \\ &= (1-t)^{-a-b} (1-2t)^{-1} f\left(\frac{x}{1-t}\right), \end{aligned} \tag{4.6}$$

using (4.6), we immediately get

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T_a^{k} \left\{\sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{b+1+n}^{n}\{f(x)\}\right\} \\ &\quad - t \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T_{a+1}^{k} \left\{\sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{b+1+n}^{n}\{f(x)\}\right\} \\ &= (1-t)^{-a-b-1} f\left(\frac{x}{1-t}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{a+b+1}^{n}\{f(x)\}. \end{aligned} \tag{4.7}$$

The relation in (4.7) leads to a generalization of Engelberg's formula (4.1). In fact, putting $f(x) =$ constant, in (4.7), we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{b+2n-2k}{n-k} \left[\frac{(a+k-n)_k}{k!} - \frac{(a+1-n+k)_{k-1}}{(k-1)!}\right] t^n = \sum_{n=0}^{\infty} \binom{a+b+n}{n} t^n,$$

and hence, comparing the coefficients of t^n, we get the identity

$$\sum_{k=0}^{n} \frac{a-n}{a-n+2k} \binom{a-n+2k}{k} \binom{b+2n-2k}{n-k} = \binom{a+b+n}{n}. \tag{4.8}$$

The result in (4.8) is a generalization of Engelberg's formula (4.1) and reduces to (4.1) for $b = 0$.

In view of (4.6), we see, from (4.7), that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T_a^{\,k} \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{b+1+n}^n \{f(x)\} \right\} \\ &\qquad - t \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{-k} T_{b+1}^k \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{a+1+n}^n \{f(x)\} \right\} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{a+b+1}^n \{f(x)\}, \end{aligned} \tag{4.9}$$

and, putting $f(x) =$ constant in (4.9), we get the identity

$$\begin{aligned} &\sum_{k=0}^{n} \left[\binom{a-n+2k-1}{k} \binom{b+2n-2k}{n-k} - \binom{b-n+2k-1}{k-1} \binom{a+2n-2k}{n-k} \right] \\ &= \binom{a+b+n}{n}. \end{aligned} \tag{4.10}$$

5. Rothe–Hagen Formula

Gould [4, 5, 8, 9, 10, 12] considered the Rothe-Hagen formula

$$\sum_{k=0}^{n} (p+qk) A_k(a,b) A_{n-k}(c,b) = \frac{p(a+c)+naq}{a+c+bn} \binom{a+c+bn}{n} \tag{5.1}$$

$$= \frac{p(a+c)+naq}{a+c} A_n(a+c,b), \tag{5.2}$$

where

$$A_k(a,b) = \frac{a}{a+bk} \binom{a+bk}{k}. \tag{5.3}$$

Gould [4, 9, 12] obtained proofs of (5.1) and posed [11, p. 52] the problem of proving (5.1) by making use of (1.3) only. Relation (1.3) can be easily seen to be a direct consequence of (1.1), and using (1.1), we proceed to establish Gould's conjecture.

In view of (5.3), we have, from (1.2), the result

$$\sum_{n=0}^{\infty} A_n(a, b)\, t^n = x \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{-n} T_{a+bn-n+1}^{n-1}\{a\} = (1 + v)^a, \tag{5.4}$$

and, differentiating with respect to t, we have, from (5.4), by using (1.1), the result

$$\sum_{n=0}^{\infty} nA_n(a, b)\, t^n = a \frac{t(1 + v)^{a+b}}{1 - (b - 1)\, v} = \frac{av(1 + v)^a}{1 - (b - 1)\, v}, \tag{5.5}$$

where $v = t(1 + v)^b$. Making use of (5.4) and (5.5), we have

$$\begin{aligned}
&\sum_{n,k=0}^{\infty} (p + qk)\, A_k(a, b)\, A_n(c, b)\, t^{n+k} \\
&= p(1 + v)^{a+c} + q \frac{av}{1 - (b - 1)\, v} (1 + v)^{a+c} \\
&= p \sum_{n=0}^{\infty} A_n(a + c, b)\, t^n + \frac{qa}{a + c} \sum_{n=0}^{\infty} nA_n(a + c, b)\, t^n \\
&= \sum_{n=0}^{\infty} \frac{p(a + c) + qan}{a + c + bn} \binom{a + c + bn}{n} t^n,
\end{aligned}$$

and hence, comparing coefficients of t^n, we get (5.1).

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